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Suppression of chaos by cyclic parametric excitation in two-dimensional maps

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Abstract. We study the qualitative change of the dynamics of a generalized two-dimensional quadratic map under the influence of parametric perturbations which operate in the chaotic parameter set. It is shown that such perturbations can lead to the suppression of chaos and appearance of a regular (periodic) behaviour. Numerically we can argue that the suppression of chaos due to the parametric excitation is caused by a shift of the windows of periodic behaviour in the bifurcation diagram.

1. Introduction

Dynamical systems which are governed by nonlinear processes show often a very complex behaviour depending on the values of their control parameters. In addition to steady-state, periodic and quasi-periodic motions, in certain regions of the parameter space such systems may possess chaotic motions. If the system exhibits a chaotic attractor then its deterministic evolution is unpredictable after a certain time. In many practical situations such a behaviour is undesirable, and should therefore be avoided. However, if it does occur or it is inevitable then specific techniques can be used to suppress chaos and to bring the system under consideration into a predictable state. There are two basic approaches to achieve a stabilization of the dynamics: the forcing method and the parametric one. In turn, each of them can be realized by including a feedback as a component of the system, which means taking the current values of dynamical variables into account. If the external perturbation is realized as a multiplicative action then the system's parameters are modified, and that is the reason why this method is called parametric. However, if a term of a certain form is added to the right-hand side of the system then the forcing method is performed.

If an external multiplicative perturbation depending on the system state is applied to achieve a required dynamics, then this technique is called parametric feedback controlling. Recently such a system has been developed to stabilize unstable periodic orbits embedded in the chaotic attractor as well as to push the system to a special target region [1–3]. Moreover, it has been verified by numerous experiments (cf [4–7]).

Furthermore, the control strategy can be independent of the present state of the system. Such a technique is now called non-feedback controlling of chaotic behaviour. The non-feedback forcing methods are now widely used as a sufficiently simple approach for the stabilization of a chaotic motion (cf [8–10]). Another way of changing the behaviour of the system without using feedback is based on a purely multiplicative action which leads to a periodic dynamics [11–14]. Such actions should be restricted to those parameter regions where the behaviour is chaotic. Recently it has been proven for chaotic one-dimensional quadratic maps and shown for a circle map which exhibits chaotic behaviour that a parametric perturbation can lead to the appearance of stable periodic orbits [15, 16]. In other words it has been argued that for a certain class of dynamical systems non-feedback control of chaotic motion is possible. It should be noted that all parameter values included in the perturbation belong to those parameter sets which ensure a chaotic dynamics [17].

Based on the latter approach the aim of this paper is to extend these results to two-dimensional maps in order to demonstrate that the concept of non-feedback parametric controlling (i.e. suppression of chaos by parametric perturbations) works not only for one-dimensional maps. First we consider a family of two-dimensional maps with external k-periodic perturbations in a general form and show that the perturbed map can be subdivided into k maps which depend on one another only via the initial conditions. Moreover, we argue that to investigate the phenomenon of chaos suppression it is sufficient to consider only one of these k maps. Thus, we expect that in certain quite general cases one can simplify the study of the perturbed two-dimensional maps. Additionally, on the basis of numerical computations we conjecture on the mechanism for the suppression of chaos in the considered two-dimensional maps. To wit, we suppose that a stabilized periodic dynamics results from a shift of the windows of periodic behaviour in the bifurcation diagram.

2. An analytical approach to investigation of chaotic two-dimensional maps with a parametric perturbation

In this section we describe quite general properties of a perturbed family of two-dimensional maps. These properties can help us to investigate numerically the phenomenon of suppression of chaos performed in section 3. So, let us consider a one-parameter family of two-dimensional maps $T_\alpha : M \to M$ in a general form:

$$T_\alpha : \boldsymbol{x} \longmapsto \boldsymbol{f}(\boldsymbol{x}, \alpha) \qquad \boldsymbol{x} \in M \tag{1}$$

where M is a compact invariant set for the map T_α, α is a control parameter, and $\boldsymbol{f} = \{f_x, f_y\}$, $\boldsymbol{x} = \{x, y\}$. Assume that α belongs to a set A of the admissible parameter values. Suppose that the map (1) can possess a chaotic dynamics. We denote the set of the corresponding parameter values by $A_c \subset A$.

For the family (1) chaotic dynamics can be established by various methods. One of them uses a widespread criterion of the positiveness of at least the largest Lyapunov exponent. Alternatively, it is possible to use the properties of a homoclinic tangency [18], mixing, or others (see, e.g. [19–21]). However, for our purposes (see section 3) it is sufficient to imply that map (1) has a positive maximum Lyapunov exponent for all parameter values $\alpha \in A_c$.

Let us introduce a parametric perturbation into map (1) as follows $G : B \to B$

$$G : \alpha \longmapsto g(\alpha) \qquad \alpha \in B \subset A. \tag{2}$$

According to the fact that chaos suppression with the help of a parametric excitation is considered, let $B = A_c$. Then the perturbed map is written in the following form

$$Q : \boldsymbol{\xi} \longmapsto \boldsymbol{q}(\boldsymbol{\xi}) \qquad \boldsymbol{\xi} \in M \times A_c \tag{3}$$

where $\boldsymbol{\xi} = (\boldsymbol{x}, \alpha)$, and $\boldsymbol{q}(\boldsymbol{\xi}) = (\boldsymbol{f}(\boldsymbol{x}, \alpha), g(\alpha))$. In other words, if the parametric perturbations (2) are included then the initial map (1) rearranges to map (3) which is three-dimensional:

$$Q:\begin{cases} x \longmapsto f_x(x,y,\alpha) \\ y \longmapsto f_y(x,y,\alpha) \\ \alpha \longmapsto g(\alpha) \qquad\qquad (x,y)\in M,\ \alpha\in A_c. \end{cases} \tag{4}$$

Thus, a projection of this map onto the plane (x, y) is map (1) with a perturbation.

Next we consider a transformation G at which map (4) can possess a regular behaviour. Let us confine our analysis to a periodic perturbation with a period $k : \alpha_{i+1} = g(\alpha_i)$, $i = 1, 2, \dots, k-1$, $\alpha_1 = g(\alpha_k)$, $\alpha_i \neq \alpha_j$ if $i \neq j$. According to our assumption, $\alpha_i \in A_c$, $1 \leqslant i \leqslant k$. Let us denote the set of all the values of α_i corresponding to the regular dynamics in (4) by A_d. Then the following two problems arise. Is the set A_d empty or not? If it is not empty then what is its Lebesgue measure? These questions are of physical significance for the following reason. If the Lebesgue measure of A_d is positive then the phenomenon of chaos suppression is observable in a real physical experiment. In other words, although owing to different kinds of external noise the parametric values α_i, $i = 1, 2, \dots, k$, are smeared, regular dynamics in map (3) should survive.

The regular behaviour of the map Q is ensured by stable periodic orbits of finite periods. In order to demonstrate the presence of order in (3), it is, therefore, necessary to find such periodic orbits. However, in such studies the following problems have to be taken into account.

(i) The period of any periodic orbit is multiple to the period of the perturbation: $\tau = nk$, where τ is the period of a periodic orbit of Q, k is the period of the perturbation G, and n is a positive integer.

(ii) In general, the projection of an orbit of period τ onto (x, y) is also an orbit of period τ. However, cases are also possible, where for some points of the map Q their coordinates coincide: $\boldsymbol{x}_i = \boldsymbol{x}_k$, $\alpha_i \neq \alpha_k$, $i \neq j$, where $(\boldsymbol{x}_i, \alpha_i)$, $(\boldsymbol{x}_k, \alpha_k)$ are points of periodic orbits of Q. Then, we do not get τ points but $(\tau - l)$ points in the plane (x, y), where l is the number of such coincidences. In particular, at $\tau = 2$ $(k = 2)$ it is possible to observe only one fixed point in the projection (x, y).

For $\tau > 2$, more complicated (exotic) situations in the projection (x, y) may occur: if a periodic solution consists of $(\tau - l)$ points then there are points in which a representative point of the map Q hits several times. Hence, such periodic motion of a two-dimensional map with a cyclic time-dependent parametric excitation cannot be called periodic orbits in a usual sense. However, the described cases of projections are degenerate ones; they are not the typical cases, and such unusual periodic orbits can be met only in specified (chosen in advance) projections of the perturbed map Q onto the plane (x, y). Therefore, periodic orbits with coincident points, as a rule, are not observed in numerical investigations.

Let us consider map (4) in the (x, y) components with a periodic perturbation. For simplicity let us assume that the map G performs a double $(k = 2)$ cyclic transformation. Then it can be written as follows

$$T = \begin{cases} T_{\alpha_1} : \boldsymbol{x} \longmapsto \boldsymbol{f}(\boldsymbol{x}, \alpha_1) \equiv \boldsymbol{f}_1(\boldsymbol{x}) \\ T_{\alpha_2} : \boldsymbol{x} \longmapsto \boldsymbol{f}(\boldsymbol{x}, \alpha_2) \equiv \boldsymbol{f}_2(\boldsymbol{x}). \end{cases} \tag{5}$$

This map may be represented via a collection of 'even' map T_1 which generates only the even numbers of iterations and 'odd' map T_2 which generates the odd iterations performed by (5):

$$\begin{aligned} T_1 &: \boldsymbol{x} \longmapsto \boldsymbol{F}_1(\boldsymbol{x}) \equiv \boldsymbol{f}_2(\boldsymbol{f}_1(\boldsymbol{x})) \\ T_2 &: \boldsymbol{x} \longmapsto \boldsymbol{F}_2(\boldsymbol{x}) \equiv \boldsymbol{f}_1(\boldsymbol{f}_2(\boldsymbol{x})). \end{aligned} \tag{6}$$

Thus, the map T (equation (5)) is a coupling of T_1 and T_2 through the initial conditions $\boldsymbol{x}_1 = \boldsymbol{f}_1(\boldsymbol{x}_0)$.

As noted above, for chaos suppression it is sufficient to find stable periodic orbits in map (5) and therefore, in map (6). In turn, map (6) consists of two consecutive transformations T_1, T_2. Therefore, any periodic orbit of period $\tau = 2n$ $(k = 2)$ of map (6) is composed by an orbit of period n of the map T_1 and an orbit of period n of the map T_2. Moreover, it is easy to show that if the function $\boldsymbol{f}_1(\boldsymbol{x})$ or $\boldsymbol{f}_2(\boldsymbol{x})$ is continuous then it is sufficient to find the stable periodic orbits only for map T_1 or T_2 [22]. In addition, it is obvious that if the perturbed map T is continuously dependent on the parameters α_1, α_2 (that is the case for several problems) then almost all values of α_1, α_2 at which the map T has a stable cycle, possess a non-zero neighbourhood in which this orbit does not disappear and conserves its stability. This suggests that the Lebesgue measure of the set A_d is positive.

For k-fold $(k > 2)$ cyclic transformation in (2), the arguments mentioned above are generalized to k maps of the following form

$$\begin{aligned}
T_1 &= \boldsymbol{f}_k(\boldsymbol{f}_{k-1}(\ldots \boldsymbol{f}_2(\boldsymbol{f}_1(\boldsymbol{x}))\ldots)) \\
T_2 &= \boldsymbol{f}_1(\boldsymbol{f}_k(\boldsymbol{f}_{k-1}(\ldots \boldsymbol{f}_3(\boldsymbol{f}_2(\boldsymbol{x}))\ldots))) \\
&\vdots \\
T_k &= \boldsymbol{f}_{k-1}(\boldsymbol{f}_{k-2}(\ldots \boldsymbol{f}_1(\boldsymbol{f}_k(\boldsymbol{x}))\ldots))
\end{aligned} \tag{7}$$

where $\boldsymbol{f}_i(\boldsymbol{x}) = \boldsymbol{f}(\boldsymbol{x}, \alpha_i)$, $i = 1, 2, \ldots, k$. Then for continuous $\boldsymbol{f}_i$ it is sufficient to find stable periodic orbits for one of the maps T_i, $i = 1, 2, \ldots, k$.

From these considerations it follows that one has to find a stable periodic orbit for the map T to achieve a suppression of chaos. However, this analytical approach does not show in which way these obtained stable periodic orbits are related to the stable periodic orbits of the system without parametric perturbation. This problem is difficult to solve analytically because in most cases one cannot compute explicitly the stable periodic orbits. Nevertheless this relationship can be explored numerically using a specific example for the analysis. This numerical investigation, presented in the next section, yields a mechanism for the appearance of the stable periodic orbits in the perturbed system as well as their relation to the stable periodic orbits in the periodic windows of the unperturbed system.

3. Numerical study of chaos suppression for a two-dimensional quadratic map

Analysing a concrete system we show numerically that the concept of the suppression of chaos by means of a cyclic parametric excitation can indeed be realized. According to section 2 we consider two-parametric transformations operating within the parameter set corresponding to chaotic motion. On the basis of these numerical computations we advance the mechanism of the transition from chaotic to regular behaviour. Let us consider as an example two quadratic one-dimensional maps which are linearly coupled in a two-dimensional map:

$$\begin{aligned}
x &\longmapsto f_x(x, y, \alpha, \gamma) = 1 - \alpha x^2 + \gamma(y - x) \\
y &\longmapsto f_y(x, y, \alpha, \gamma) = 1 - \alpha y^2 + \gamma(x - y)
\end{aligned} \tag{8}$$

where α and γ are the control parameters. This system has been studied intensively by several authors (cf [23, 24]). Therefore, we can omit a lot of details in the bifurcation behaviour of system (8) and investigate the possibility of the suppression of chaos only.

3.1. General properties of the map

First of all, note the general properties of map (8). It is not hard to see that it is symmetric with respect to the change of the variables $x \to y$. This means that if a point $(\tilde{x}, \tilde{y})$ is a solution of the fixed point equation then a point $(\tilde{y}, \tilde{x})$ is also a solution of the same equation. In addition, if an initial point (x_0, y_0) of map (8) belongs to a line $x = y$, i.e. $x_0 = y_0$, then all future iterations will also belong to this line. In other words, for $x_0 = y_0$ and any $n \geqslant 0$, $x_n = y_n$. Therefore, in this case map (8) degenerates into the well known one-dimensional quadratic map.

One can easily find fixed points in (8). Due to the symmetry of map (8), two different cases can be distinguished, $x = y$ and $x \neq y$. To find period-2 orbits of map (8), we analyse the equations for the second iteration:

$$\begin{aligned} x &= 1 - \alpha(1 - \alpha x^2 + \gamma(y - x))^2 + \gamma(x - y)[\alpha(x + y) + 2\gamma] \\ y &= 1 - \alpha(1 - \alpha y^2 + \gamma(x - y))^2 + \gamma(y - x)[\alpha(x + y) + 2\gamma]. \end{aligned} \tag{9}$$

Once more there are the two different cases with $x = y$ and $x \neq y$. We only present the solution for the second case ($x \neq y$) which is used as a starting point for our numerical study.

Introducing simplified notation one can obtain the values of the periodic points:

$$\begin{aligned} \tilde{x}_{1,2} &= \frac{p - 2\gamma}{2\alpha} \pm \frac{\sqrt{4\delta - (p - 2\gamma)^2 - 2/p}}{2\alpha} \\ \tilde{y}_{1,2} &= \frac{p - 2\gamma}{2\alpha} \mp \frac{\sqrt{4\delta - (p - 2\gamma)^2 - 2/p}}{2\alpha} \end{aligned} \tag{10}$$

where $\delta = \alpha + \gamma$ and p denotes one of the following six values: $p_1 = 1$, $p_2 = -1$, $p_{3,4} = \gamma - \frac{1}{2}\sqrt{1 + 4\delta} \pm \frac{1}{2}\sqrt{(2\gamma - \sqrt{1 + 4\delta})^2 - 4}$, $p_{5,6} = \gamma + \frac{1}{2}\sqrt{1 + 4\delta} \pm \frac{1}{2}\sqrt{(2\gamma + \sqrt{1 + 4\delta})^2 - 4}$. Obviously, for $p = p_2$ the points $(\tilde{x}_{1,2}, \tilde{y}_{1,2})$ are the fixed points. Consequently, for the other $p = p_i$, $i = 1, 3, 4, 5, 6$, there exist pair-wise points which create period-2 orbits in map (8). The stability of all these fixed points and period-2 orbits can be easily checked numerically.

3.2. Bifurcations of the unperturbed map

Our numerical treatment of the behaviour of map (8) mainly follows [23]. Figure 1 shows part of the bifurcation diagram. For the sake of simplicity, it contains only those bifurcation lines which are taken into account in the following computations. We start our analysis with the asymmetric period-2 orbit given by (10) for $p = p_2 = -1$. It is stable below the first bifurcation line in figure 1 (region A). To explain briefly the route to chaos we describe the bifurcations along a vertical line with fixed γ in figure 1 (i.e. $\gamma = 0.27$). The period-2 solution loses its stability at a bifurcation point where two complex conjugate eigenvalues cross the unit circle and a quasi-periodic motion occurs which corresponds to the existence of an invariant curve in the state space. This bifurcation is called bifurcation generating an invariant curve throughout this paper. Note that this kind of bifurcation resembles the appearance of an invariant curve in the Poincaré map for time-continuous systems which corresponds to the bifurcation of a periodic orbit into an invariant torus.

In the parameter region studied the quasi-periodic behaviour yields two invariant curves in the (x, y) plane which are symmetric with respect to the line $x = y$ (region B). An increase in the parameter α leads to a frequency locking of the ratio 2/5. This frequency-locking domain (region C) is bounded by two curves which form a well known Arnol'd

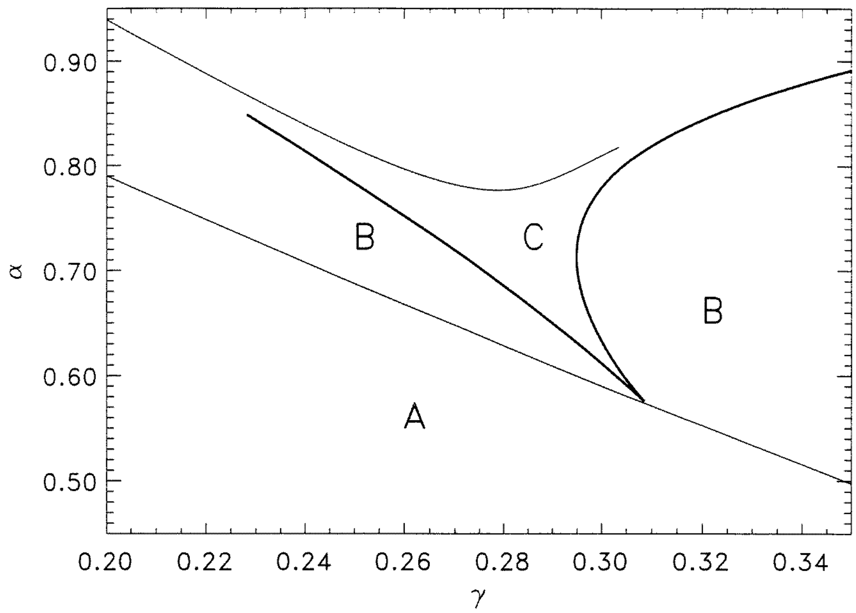

Figure 1. Part of the bifurcation diagram of the linearly coupled quadratic maps in the neighbourhood of the resonance 2/5: A, period-2 orbit; B, invariant curve; C, resonance 2/5, period-10 orbit.

tongue. Inside this tongue an orbit of period 10 arises. With a further increase in α this period-10 orbit loses its stability again via bifurcation generating an invariant curve. The resulting attractor consists of 10 invariant curves. This scenario is completely repeated until the phase-locked periodic orbit undergoes a period doubling. Chaotic behaviour appears in map (8) by means of a period-doubling cascade or the destruction of the invariant curves.

For the sake of simplicity, we keep one of the parameters fixed during further study. In the model system considered the parameter α describes the nonlinearity of the dynamics inside each system, whereas γ is a coupling parameter. One can assume that α is an intrinsic parameter of the dynamics which is not accessible for control purposes. Therefore, we assume that only the coupling γ can be varied. But the described method is independent of that choice.

To obtain chaotic behaviour, α has to be fixed above the last bifurcation line in figure 1. For our computations we have chosen $\alpha = 0.85$. To determine a set of parameter values γ with $\alpha =$ constant which corresponds to chaotic behaviour of (8), the Lyapunov exponents have been calculated (figure 2). Let us denote the set of the parameter values γ, for which the maximum Lyapunov exponent is positive, by S_α. This set S_α (for a fixed α) has the same meaning as A_c for map (1) in section 2. Additionally, within the range of parameter values of chaotic behaviour we find several intervals corresponding to periodic orbits with different periods. These intervals are indicated by a negative maximum Lyapunov exponent.

3.3. Suppression of chaos in the perturbed system

Now, applying the approach developed in the previous section, we introduce a parametric perturbation of system (8) in the form of a periodic variation (switching) of the coupling constant γ for $\alpha =$ constant. In other words, we consider a k-periodic transformation

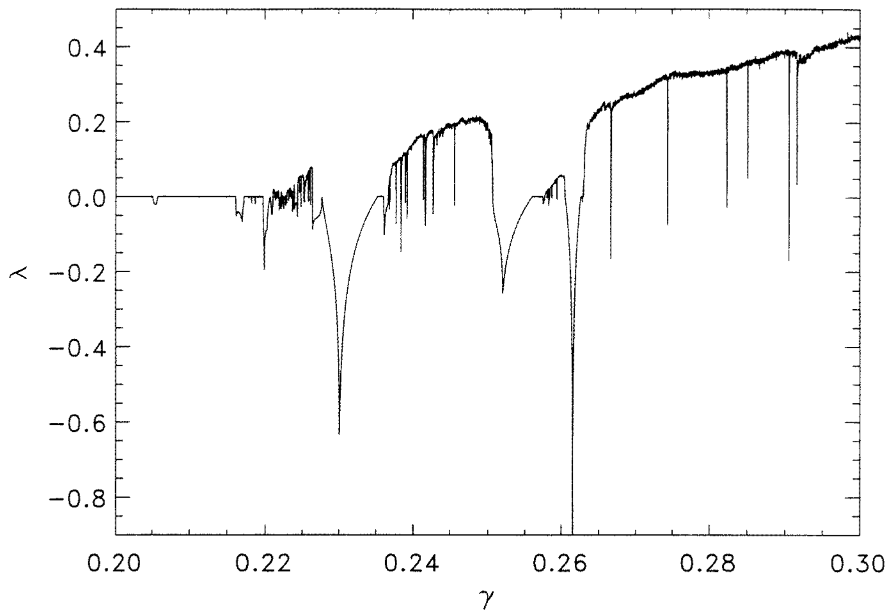

Figure 2. The maximum Lyapunov exponent versus γ for system (8) ($\alpha = 0.85$).

$\gamma_{i+1} = g(\gamma_i)$, $i = 1, 2, \ldots, k-1$, $\gamma_1 = g(\gamma_k)$, $\gamma_i \neq \gamma_j$ if $i \neq j$. Then, in the simplest case of a *two*-periodic variation (2), $k = 2$, the investigated map (8) is a map consisting of *two* sequential steps: in the first step the map (8) depends on the parameter value γ_1 and, in the second step, it depends on the value γ_2. In terms of iterations it looks as follows

$$\begin{aligned} x_{2n+1} &= f_{1,x}(x_{2n}, y_{2n}, \alpha, \gamma_1) = 1 - \alpha x_{2n}^2 + \gamma_1(y_{2n} - x_{2n}) \\ y_{2n+1} &= f_{1,y}(x_{2n}, y_{2n}, \alpha, \gamma_1) = 1 - \alpha y_{2n}^2 + \gamma_1(x_{2n} - y_{2n}) \\ x_{2n+2} &= f_{2,x}(x_{2n+1}, y_{2n+1}, \alpha, \gamma_2) = 1 - \alpha x_{2n+1}^2 + \gamma_2(y_{2n+1} - x_{2n+1}) \\ y_{2n+2} &= f_{2,y}(x_{2n+1}, y_{2n+1}, \alpha, \gamma_2) = 1 - \alpha y_{2n+1}^2 + \gamma_2(x_{2n+1} - y_{2n+1}). \end{aligned} \tag{11}$$

Here, in order to avoid the calculations of the function g we use explicitly γ_1 and γ_2.

Suppose that the parameter values in (11) satisfy the conditions $\gamma_1, \gamma_2 \in S_\alpha$. Our aim is to find a certain pair $\gamma_{1,2} \in S_\alpha$, $\alpha =$ constant, for which the chaotic dynamics of the unperturbed map is stabilized and becomes periodic. However, the following points should be noted. In studies of the perturbed system (11) a problem arises concerning its comparison with the unperturbed one (8), since system (11) depends on two variable parameters γ_1, γ_2 ($\alpha =$ constant) while in system (8) the parameter γ is fixed at the same value along all iterations. For our case one can compare the perturbed system with such an unperturbed one in which the parameter γ is equal to one of the values γ_1 or γ_2. Let us agree on the condition that the unperturbed system is the system with $\gamma = \gamma_1$. This agreement holds from now on, and all results should be considered only in this sense.

For a more convenient and instructive study of the arising behaviour, let us introduce a new parameter Γ as the amplitude of the parametric perturbation

$$\Gamma = \gamma_2 - \gamma_1. \tag{12}$$

According to the obvious supposition that only small changes in the parameter values are admissible, the amplitude should vary over a small range only.

Two methods are possible to study the influence of a periodic parametric excitation.

(i) One can calculate the changes in the bifurcation diagram and the Lyapunov exponents for an increasing amplitude Γ of the cyclic excitation. But it is important to note that in the case of a uniformly increasing amplitude one has to be careful with the interpretation of the results because both values γ_1 and γ_2 may not belong to the chaotic region S_α for every Γ as is required.

(ii) The influence of a parametric excitation with a fixed amplitude can be studied. In this case it is easy to guarantee that both γ_1 and γ_2 are elements of S_α.

To ensure that the parametric excitation does not change drastically the whole qualitative behaviour, let us first look at the bifurcation phenomena of the perturbed system (11) in comparison to the unperturbed one (8). We indeed find the same route to chaos, but all occurring bifurcation points are shifted towards other values of α with increasing amplitude Γ. To demonstrate this shift, let us focus on the first bifurcations which are obtained along the line $\gamma = 0.27$ in figure 1. All three bifurcations are taken into account, namely the bifurcation, where the period-2 orbit loses its stability and two invariant curves arise, the point of frequency-locking which is a saddle-node bifurcation for the period-10 orbit occurring inside the resonance region, and the bifurcation generating 10 invariant curves from the period-10 orbit. Figure 3 shows that the first two bifurcation points are shifted towards lower values of α with increasing amplitude Γ whereas the last bifurcation is shifted towards higher values of α. This shift of bifurcation points can also be presented by fixing the amplitude Γ of the periodic excitation and comparing the two calculated bifurcation diagrams (figure 4). Therefore, we can conclude that the small parametric perturbation yields only a small shift of bifurcations. However, essential changes in the bifurcation behaviour, such as the appearance of new bifurcations, are not observed. It is important to

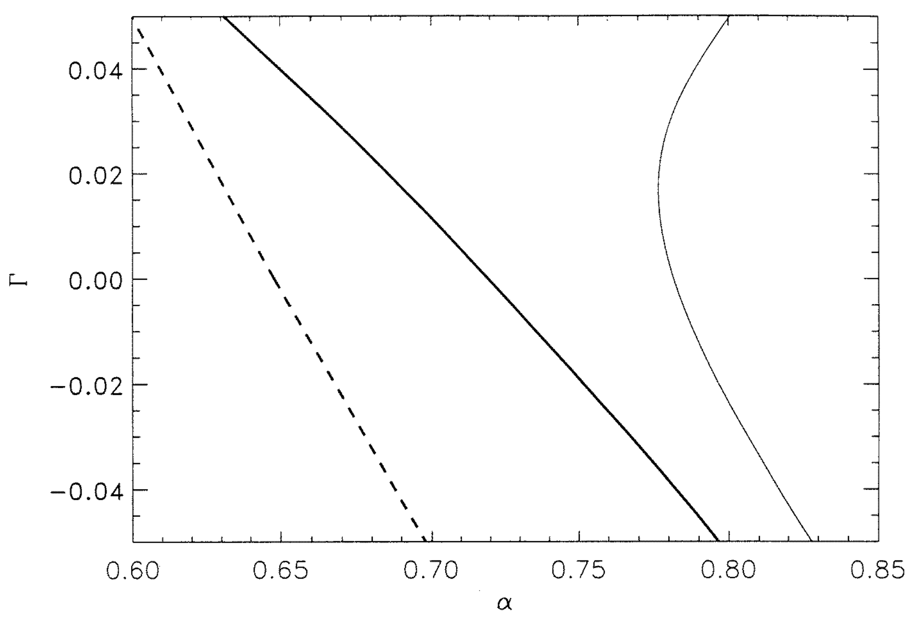

Figure 3. The shift of the bifurcation points shown in figure 1 versus the amplitude Γ of the parametric periodic excitation at $\gamma_1 = 0.27$: broken curve, bifurcation generating two invariant curves from the period-2 orbit; bold full curve, saddle-node bifurcation of the period-10 orbit; thin full curve, bifurcation generating 10 invariant curves from the period-10 orbit.

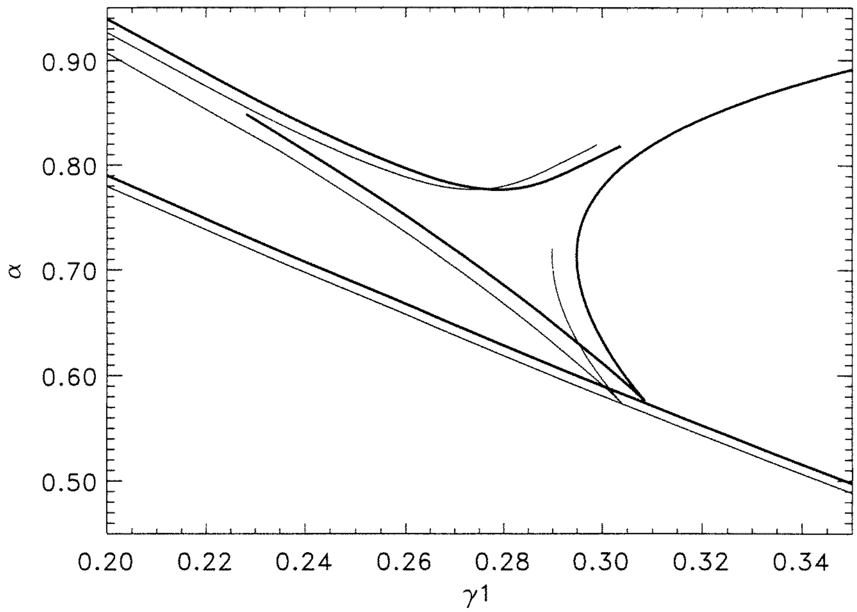

Figure 4. Bifurcation diagram of the coupled quadratic maps with parametric excitation (11) in comparison to the unperturbed system (8): bold full curve, without parametric excitation, $\Gamma = 0$; thin full curve, $\Gamma = 0.01$.

note that the direction of this shift is different for each specific bifurcation point in each model system; it cannot be predicted in advance just by analytical means. In particular, there is no general relationship between the sign of the amplitude Γ and the direction of the shift.

Let us turn to the discussion of the chaotic motion in the perturbed system. As already mentioned, the Lyapunov exponents have been computed to decide whether some set of parameter values belongs to the chaotic region S_α or not. However, referring to the dependence of the Lyapunov exponents versus the parameter γ_1 for a fixed amplitude $\Gamma = 0.01$, one can observe just the same influence of the parametric perturbation. Comparing figure 2 with figure 5, we notice that the shape of the dependence remains qualitatively almost the same and a certain shift of the intervals of the periodic behaviour ('periodic windows' which correspond to intervals with a negative maximum Lyapunov exponent) towards lower values of γ_1 is observed (figure 5). As in the unperturbed system intervals with orbits of different periods can be found.

According to our aim to study the suppression of chaos using a periodic parametric perturbation, let us look at stable periodic motions which can be found within the chaotic region in figure 5. The parameter α is again kept fixed at $\alpha = 0.85$. Let us choose two parameter values γ_1 and γ_2 which are separated by a value $\Gamma = 0.01$ and which both have to be elements of S_α in the unperturbed system (equations (8)). As an example, we take the value $\gamma_1 = 0.269\,14$ and $\gamma_2 = 0.279\,14$ which fulfil the above suppositions (cf figure 2). The iteration of system (11) with these parameter values yields a stable period-14 orbit which corresponds to the first small periodic interval with $\gamma_1 > 0.26$ seen in figure 5. To make the comparison of the perturbed system with the unperturbed one easier, a small part of figures 2 and 5 is presented in one graph (figure 6). Figure 6 shows that a stable periodic motion can occur in the parametric excited system at $\gamma_1 = 0.269\,14$, $\Gamma = 0.01$, although

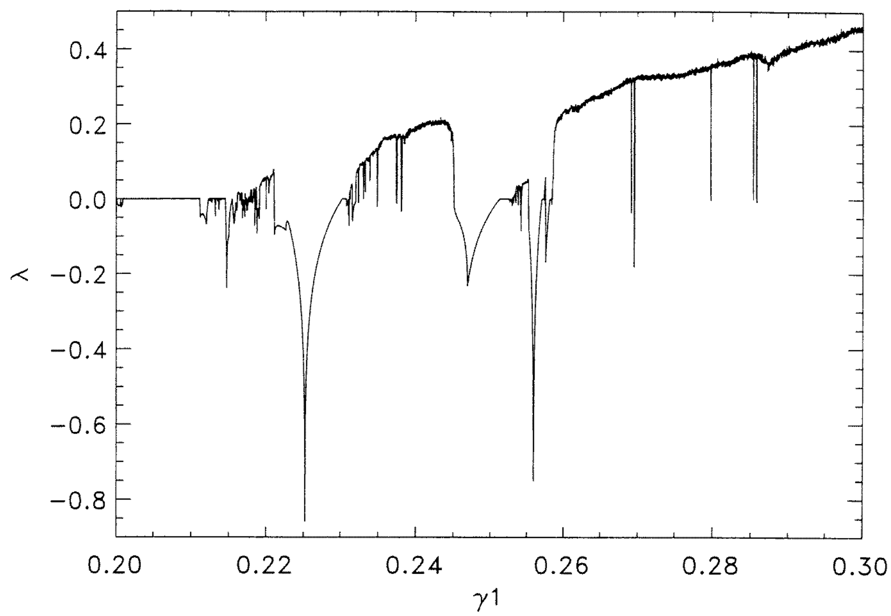

Figure 5. The maximum Lyapunov exponent versus γ_1 for system (11) ($\alpha = 0.85$, $\Gamma = 0.01$).

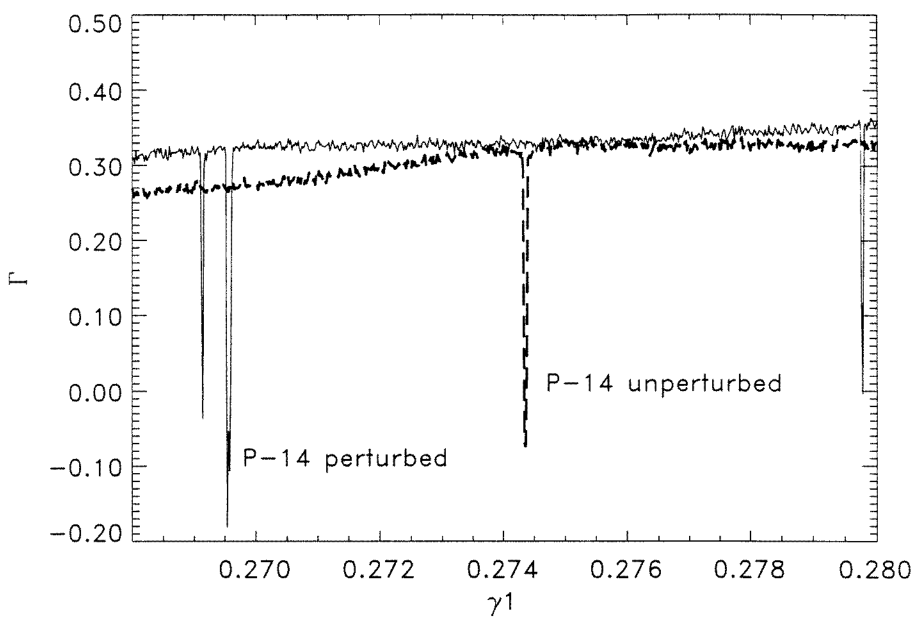

Figure 6. Enlargement from figures 2 and 5: thin full curve, $\Gamma = 0.01$; bold broken curve, $\Gamma = 0$.

for the unperturbed system ($\Gamma = 0$) the motion is chaotic (for $\gamma = \gamma_1 = 0.269\,14$ as well as for $\gamma = \gamma_2 = 0.279\,14$). This numerical result demonstrates that chaos can be suppressed by a parametric excitation with a certain amplitude even if the values for the excitation lie in the chaotic parameter range.

3.4. A possible mechanism for chaos suppression

The theory outlined in section 2 does not establish the mechanism for the appearance of a stable periodic motion. This part is addressed to this question. The shift of the bifurcation diagram, which has been discussed above, plays the key role in the stabilization of motion. Not only the values of the bifurcations along the route to chaos are shifted but also the structure of the chaotic parameter region itself changes under the influence of the periodic perturbation. It is an important fact that the range of the investigated parameter values contains a lot of intervals which correspond to windows of periodic behaviour of map (11) as well as of map (8). This means that the chaotic and regular dynamics in maps (8) and (11), respectively, are closely interwoven. It can be shown numerically that these intervals with periodic behaviour in map (8) are shifted due to the parametric perturbation to other parameter intervals. This shift depends on the amplitude of the excitation.

To investigate the relation between the periodic intervals in the unperturbed and the perturbed system in more detail, the appearance and disappearance of the periodic motion is studied. It is known that periodic orbits in windows occur due to a saddle-node bifurcation. Due to the variation of the parameter value this periodic orbit undergoes a period-doubling cascade and ends up in chaos again resulting in the disappearance of the periodic behaviour. Let us analyse the bifurcation structure of the periodic orbit observed in the perturbed system. For this purpose we consider the arising period-14 orbit found at $\alpha = 0.85$, $\gamma_1 = 0.269\,14$, $\gamma_2 = 0.279\,14$ depending on the amplitude Γ. One can see from figure 7 that there is only a small interval of amplitudes where this stable period-14 orbit exists. This interval is

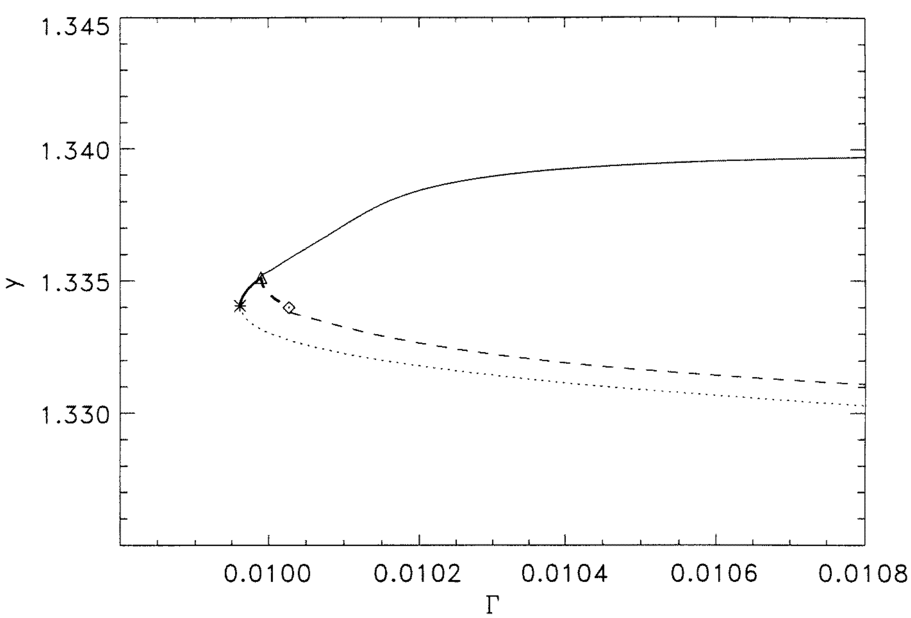

Figure 7. Stability of the period-14 orbit depending on the amplitude of the parametric perturbation Γ ($\alpha = 0.85$, $\gamma_1 = 0.269\,14$): bold full curve, stable period-14 cycle; thin full curve and thin dotted line, unstable period-14 cycle; bold broken curve, stable period-28 cycle; thin broken curve, unstable period-28 cycle; star, saddle-node bifurcation for period-14 orbit ('birth' of the stable period-14 orbit); triangle, period doubling period-14 to period-28; diamond, period doubling period-28 to period-56. All other periodic orbits with period-56 and higher have not been continued and are, therefore, omitted.

bounded on one side by a saddle-node bifurcation. This bifurcation occurs at an amplitude of $\Gamma = \Gamma_c = 0.009\,961$, when the stable and unstable period-14 orbits merge and disappear. For the lower values of Γ, $\Gamma < \Gamma_c$, up to $\Gamma = 0$ ($\gamma_2 = \gamma_1$), i.e. for the unperturbed system, the period-14 orbit does not exist for the given $\alpha = 0.85$ and $\gamma_1 = \gamma_2 = 0.269\,14$. With increasing amplitude a period-doubling cascade, which ends up in chaos again, is observed. This transition to chaos is the second boundary of the stable periodic motion in the perturbed system.

Let us turn to the question whether this stable periodic orbit is somewhat connected to the orbits occurring in the unperturbed system. For this purpose it is sufficient to show that the point of 'birth' of the stable periodic orbit (in our case the saddle-node bifurcation in figure 7) can be continued to $\Gamma = 0$ by variation of γ_1 for fixed α. If such a continuation is possible then the periodic window in the perturbed system is related to a periodic window of the same period in the unperturbed system. Using path-following methods [25] we find the appearance of the period-14 window in the unperturbed system. The possibility of such a continuation illustrates again the shift of the periodic window. In our example the saddle-node bifurcation of the period-14 orbit for $\Gamma = 0$ can be observed at $\gamma_1 \cong 0.2745$. Therefore, one should find at this value of γ_1 a periodic behaviour in the unperturbed system. To this end we refer to the graph of the maximum Lyapunov exponents (figure 6), where the saddle-node bifurcation corresponds approximately to this value of γ_1 where the periodic window arises; this means that the maximum Lyapunov exponent crosses the zero line towards negative values. Indeed, one can recognize in figure 6 this 'birth' of the stable period-14 orbit for both systems: at $\gamma_1 \approx 0.2691$ and $\Gamma = 0.01$ (thin full curve) for the perturbed system and at $\gamma_1 \approx 0.274$ and $\Gamma = 0$ (bold broken curve) for the unperturbed one. Hence, the parameter value for the appearance of the period-14 window obtained from the continuation of the saddle-node bifurcation coincides with the results from the computation of the maximum Lyapunov exponent.

4. Conclusion

We discuss here two-dimensional maps with chaotic behaviour and the possibility of chaos suppression by means of parametric perturbations. From the analytical approach we can make the following useful inference. If we perturb a chaotic map by means of a k-cyclic parametric transformation then one can separate the obtained perturbed map into k independent maps (except for the initial conditions). These can be constructed from the initial (unperturbed) map with the help of sequential permutation (7). In addition, it is sufficient to consider only one of k independent maps to determine the type of the dynamics of the whole perturbed map. This fact can essentially simplify the investigations of the maps under parametric perturbations. Thus, the theoretical considerations give, in principle, a key to the analytical study of the problem of chaos suppression for periodically excited chaotic maps and allows one to find the necessary parameter values at which the chaotic behaviour can be stabilized. Moreover, these parameter values have some (small) neighbourhood in which the behaviour remains stable. In other words, in a physical experiment this regular dynamics should be observable and should not be destroyed due to different kinds of small enough external noise which smear the required parameter values.

Another conclusion which follows from the performed analysis concerns functionally coupled maps with chaotic dynamics. If we construct a composition of k identical maps and each of them exhibits chaotic properties, then the resulting chain of coupled maps can have regular behaviour. Therefore, in a certain sense the deterministic chaos can be suppressed by means of the deterministic chaos.

In turn, as numerical investigations have shown, the induced regular behaviour may be apparently explained as a shift in the bifurcation diagram of the unperturbed system. Within the chaotic region the parametric excitation leads to stable periodic motions due to a shift of periodic windows. We would like to emphasize that the chosen parameter values which have to be elements of S_α are always in some neighbourhood of a periodic window, since the windows are closely interwoven with areas of chaotic dynamics. Therefore, the shift of the windows due to the parametric excitation pushes the system into a periodic motion in such a window. This can be regarded as the main mechanism leading to stable periodic motions using parametric excitation within the chaotic region.

This allows us to suppose that the control of chaotic behaviour without feedback for systems with such types of windows can be transformed into a goal-oriented method where a periodic orbit chosen in advance can be stabilized. In a practical approach for a specific model system one has first to choose a periodic window in the unperturbed system. In the next step one has to determine the necessary amplitude of the perturbation using path-following methods for the continuation of the saddle-node bifurcation, the 'birth', of the desired periodic orbit. Knowing the amplitude of the parametric perturbation, one can stabilize the chaotic behaviour and turn the motion into the desired periodic one. However, it is important to note that for such a goal-oriented non-feedback control it is necessary to have an explicit model for the system in the form of equations to perform the numerical calculations needed for the determination of the appropriate amplitude of the parametric perturbation. It is an open question as to how to generalize this method to systems which are given only by measurements, and it needs further investigation. However, due to the small width of most of the periodic windows one may find only the largest ones numerically. At least further analytical investigation should clarify this problem.

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